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Twisted Poincaré Invariant Quantum Field Theories

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Abstract: It is by now well known that the Poincaré group acts on the Moyal plane with a twisted coproduct. Poincaré invariant classical field theories can be formulated for this twisted coproduct. In this paper we systematically study such a twisted Poincaré action in quantum theories on the Moyal plane. We develop quantum field theories invariant under the twisted action from the representations of the Poincaré group, ensuring also the invariance of the S-matrix under the twisted action of the group . A significant new contribution here is the construction of the Poincaré generators using quantum fields.

Contents

1. Introduction

Recently it was pointed out that the apparent lack of Poincaré symmetry on the Groenewald-Moyal (GM) plane can be given a new interpretation, using the known results in quantum group theory, according to which the Poincaré symmetry is still preserved, though with a new coproduct [1–3]. For example, consider the following integral on the GM plane:

$$S[\phi] = \int d^d x\, \phi(x) * \phi(x) * \cdots * \phi(x), \tag{1.0.1}$$

$$\phi * \phi(x) = \phi\, e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu}\, \phi(x), \tag{1.0.2}$$

where ϕ is a scalar field and the $*-$product as defined in (1.0.2) is used to multiply functions on the GM plane. The fields carry a representation of the Poincaré group and transform under a Poincaré tranformation g in the usual way:

$$g \triangleright \phi(x) = \phi(g^{-1}x)\ . \tag{1.0.3}$$

If we treat the product $\phi * \phi * \cdots * \phi$ again as a scalar function and transform it in the usual way, the integral S will be invariant. But the problem is that if we transform the fields individually in (1.0.1), then

$$g \triangleright \phi \, * \, g \triangleright \phi \, * \cdots * \, g \triangleright \phi \neq g \triangleright (\, \phi * \phi * \cdots * \phi \,). \tag{1.0.4}$$

Hence S is not Poincaré invariant in the sense that

$$S[g \triangleright \phi] \neq \; S[\phi] \; . \tag{1.0.5}$$

But the group action on individual fields in a product comes from a coproduct Δ_0 on the group. The non-invariance of S occurs because the usual coproduct Δ_0 on the Poincaré group,

$$\Delta_0(g) = g \otimes g$$

is not compatible with $*-$multiplication.

Now there is a well defined way of deforming the coproduct so that the deformed coproduct Δ_θ gives us the right hand side in (1.0.4), that is,

$$m_\theta \, \Delta_\theta(g) \, f \otimes g \;\; = \;\; g \triangleright m_\theta \, (f \otimes g) \tag{1.0.6}$$

where m_θ is the $*$-multiplication map:

$$m_\theta(f \otimes g) = f * g \; . \tag{1.0.7}$$

We recall this coproduct in Section 2.

The invariance of classical field theories for the Poincaré group action with the twisted coproduct does not automatically mean that quantum theories associated with such classical field theories will also be invariant under the twisted Poincaré transformations. For example, the Poincaré invariance of the measure used to define the functional integral has to be reconsidered [4, 5].

In this paper we take the route of Hamiltonian quantum theory to study quantum theories. We examine the construction of quantum field theories with Poincaré invariance with the twisted coproduct (for the Poincaré group) as the symmetry principle. Our treatment closely follows the general theory of quantum group symmetries in quantum mechanics as discussed by Mack and Schomerus [6, 7].

The paper is organized as follows. We briefly discuss the Drinfel'd twist of the coproduct of the Poincaré group in Section 2. Section 3 reviews the statistical properties of the Hilbert space of a quantum theory with the Hopf algebra associated with the twisted coproduct as its symmetry and its well-known connection with the $\mathcal{R}$ matrix of the Hopf algebra is explained. In section 4 explicit expressions for the quantum generators for the Lie algebra of the Poincaré group (with the twisted coproduct) are given in terms of creation and annihilation operators . Section 5 discusses the form of interaction Hamiltonians which give us Lorentz invariant S-matrices. Section 6 discusses some conventions in the definition of creation/annihilation operators. Section 7 concludes the paper.

2. The Twisted Coproduct of the Poincaré Group

For completeness and fixing the notation, we briefly recall the Drinfel'd twist of the coproduct on the Poincaré group. For details see [3].

The usual Poincaré group P has associated with it a canonical coproduct Δ_0,

$$\Delta_0(g) \;=\; g \otimes g\;, \quad g \in P \tag{2.0.1}$$

or at the Lie algebra level, for u in the Lie algebra $\mathcal{P}$,

$$\Delta_0(u) \;=\; u \otimes \mathbb{1} + \mathbb{1} \otimes u. \tag{2.0.2}$$

These definitions extend to the group algebra $\mathfrak{P}$ of P and the universal enveloping algebra $\mathscr{P}$ of $\mathcal{P}$ by linearity. The group algebra $\mathfrak{P}$ and the universal enveloping algebra $\mathscr{P}$ have the full Hopf algebra structure with the following counit and antipode defined on the pure group elements g of $\mathfrak{P}$ by

$$\epsilon(g) \;=\; \mathbb{1} \quad , \quad S(g) \;=\; g^{-1} \tag{2.0.3}$$

which is then extended by linearity to the whole $\mathfrak{P}$. Alternatively on the generators u and $\mathbb{1}$ of $\mathscr{P}$, ϵ and S read

$$\begin{aligned} \epsilon(u) \;&=\; 0 \quad , \quad \epsilon(\mathbb{1}) \;=\; \mathbb{1} \\ S(u) \;&=\; -u \quad , \quad S(\mathbb{1}) \;=\; \mathbb{1}\;. \end{aligned} \tag{2.0.4}$$

They are then defined on all of $\mathscr{P}$ by linearity.

We can think of the elements of the universal enveloping algebra $\mathscr{P}$ as living in the group algebra $\mathfrak{P}$ and vice versa. Notice that we can either define the above structures on the pure group elements and then by linearity and limiting procedures, this defines them on the Lie algebra, or we can have the definitions of Δ, S and ϵ on the Lie algebra elements and by linearity they induce the coproduct etc. on pure group elements. So only one of the definitions in (2.0.4, 2.0.3) is really needed.

We can define a new coproduct Δ_θ on $\mathfrak{P}$ and $\mathscr{P}$ by the Drinfel'd twist:

$$\Delta_\theta(g) \;=\; F_\theta^{-1}\,\Delta_0(g)\,F_\theta \tag{2.0.5}$$

$$\Delta_\theta(u) \;=\; F_\theta^{-1}\,\Delta_0(u)\,F_\theta, \tag{2.0.6}$$

where

$$F_\theta \;=\; e^{-\frac{i}{2}\theta_{\mu\nu}P^\mu\otimes P^\nu} \quad , \quad F_\theta^{-1} \;=\; e^{\frac{i}{2}\theta_{\mu\nu}P^\mu\otimes P^\nu} \tag{2.0.7}$$

where P^μ is momentum operator.

With this coproduct (and the same counit and antipode as before), we obtain the twisted Hopf algebra of the Poincaré group.

The Poincaré group with the twisted coproduct acts on the algebra $\mathcal{A}_\theta$ of functions $f \in \mathbb{R}^d$ with the product defined through a $*-$product, compatibly in the sense of eq. (1.0.6). The $*-$product is given by

$$m_\theta(\phi\otimes\psi) = m_0\mathcal{F}_\theta\,\phi\otimes\psi. \tag{2.0.8}$$

where $\mathcal{F}_\theta$ is the differential operator representing F_θ on the space of functions and m_0 is the usual untwisted multiplication map.

3. Twisted Coproduct and the Hilbert Space

Here we review the construction of the Hilbert space with the twisted Hopf-Poincaré symmetry, using the language of Mack and Schomerus. The general idea of twisted symmetries in the quantum Hilbert space is given in [6,7], while most of the results of this section have been given in [4].

3.1 Transformation Law for Hilbert Space

The Hilbert space of quantum fields consists of all multi-particle states. We start with the single particle states.

3.1.1 Single Particle States

As usual we identify the single particle states with the one-particle irreducible representations of the (identity component of) Poincaré group. For simplicity, we consider the massive spinless case so that a basis of the states can be labeled just by momenta.

We choose the following normalization for the single particle states:

$$\begin{aligned} \langle k|p\rangle \;&=\; 2k^0\delta^3(k-p)\ , \\ k^0 &= \sqrt{\vec{k}^2+m^2}\ , \\ m &= \text{mass of the particle.} \end{aligned} \tag{3.1.1}$$

We have unitary operators $U(g)$ on the Hilbert space which form a representation of the Poincaré group $P_+^\uparrow$. On single particle states, they act as usual,

$$U(g)|k\rangle \;=\; |gk\rangle \quad ,\ g\in P \tag{3.1.2}$$

and hence

$$U(g_1)U(g_2) \;=\; U(g_1g_2) \qquad \text{on single particle states.} \tag{3.1.3}$$

We can also write eq (3.1.2) as

$$U(g)|k\rangle \;=\; \int \frac{d^3k'}{2k'^0}\,\rho_{k'k}(g)|k'\rangle \tag{3.1.4}$$

where

$$\begin{aligned} \rho_{k'k}(g) &= \langle k'|U(g)|k\rangle \\ &= 2k'^0\delta^3(k'-gk). \end{aligned} \tag{3.1.5}$$

(The integral(sum) will always be assumed over the repeated indices, unless otherwise stated, though we will not explicitly write it.) $\rho_{k'k}(g)$ are the matrix elements of $U(g)$ and form a representation of the group:

$$\rho_{kp}(g_1)\rho_{pk'}(g_2) \;=\; \rho_{kk'}(g_1g_2). \tag{3.1.6}$$

The representation of the group extends naturally to the group algebra by linearity. For $f = \sum_i a_i g_i \in \mathfrak{P}$ and $g_i \in P$

$$U(f) = \sum_i a_i U(g_i) \ . \tag{3.1.7}$$

(More generally we should write $f = \int dg\, f(g)\, g$ and $U(f) = \int dg\, f(g)\, U(g)$, where dg is the Haar measure on the Poincaré group, but it does not change any of the arguments.)

Again we can write

$$U(f)|k\rangle \;=\; \rho_{k'k}(f)|k'\rangle \tag{3.1.8}$$

where

$$\rho(f) \;=\; \sum_i a_i \rho(g_i) \ . \tag{3.1.9}$$

It is straightforward to check that $U(f)$ and the matrix $\rho(f)$ form a representation of $\mathfrak{P}$.

Here it is worth noting that we define the vacuum to be invariant under the group so that under a pure group element,

$$U(g)|0\rangle \;=\; |0\rangle \tag{3.1.10}$$

but under a general group algebra element f,

$$\begin{aligned} U(f)|0\rangle &= \sum_i a_i U(g_i)|0\rangle \\ &= \sum_i a_i |0\rangle \\ &= \epsilon(f)|0\rangle \ . \end{aligned} \tag{3.1.11}$$

The single particle sector is unaffected by the twist. The twist has only changed the coproduct and the coproduct does not show up in the single particle representations.

3.1.2 Multi-Particle States

Let us for simplicity first consider the two-particle sector. The two-particle states carry two momentum indices and hence transform according to the two-fold tensor product of single particle representations. The tensor products of representations are defined using the coproduct. In the undeformed case, the coproduct over the group is given by eq (2.0.1), so we have the familiar transformation, for a pure group element g,

$$\begin{aligned} U(g)|k_1, k_2\rangle \;&=\; \rho_{k_1'k_1} \otimes \rho_{k_2'k_2} \left(\Delta_0(g)\right) |k_1', k_2'\rangle && (3.1.12) \\ &=\; \rho_{k_1'k_1} \otimes \rho_{k_2'k_2} \left(g \otimes g\right) |k_1', k_2'\rangle && (3.1.13) \\ &=\; \rho_{k_1'k_1}(g)\, \rho_{k_2'k_2}(g)\, |k_1', k_2'\rangle && (3.1.14) \\ &=\; |gk_1, gk_2\rangle \ . && (3.1.15) \end{aligned}$$

But in the twisted case, we must use the deformed coproduct Δ_θ, so that

$$
\begin{aligned}
U(g)|k_1, k_2\rangle &= \rho_{k'_1 k_1} \otimes \rho_{k'_2 k_2} (\Delta_\theta(g)) \, |k'_1, k'_2\rangle \\
&= \rho_{k'_1 k_1} \otimes \rho_{k'_2 k_2} \, (F_\theta^{-1} \, (g \otimes g) \, F_\theta) \, |k'_1, k'_2\rangle \\
&= e^{-\frac{i}{2}\theta_{\mu\nu} k_1^\mu k_2^\nu} \, e^{\frac{i}{2}\theta_{\mu\nu}(gk_1)^\mu (gk_2)^\nu} \, |gk_1, gk_2\rangle.
\end{aligned} \tag{3.1.16}
$$

Similarly for an n-particle state we have the transformation law

$$
\begin{aligned}
&U(g)|k_1\,,\, k_2\,,\, \cdots\, k_n\rangle \;= \\
&\rho_{k'_1 k_1} \otimes \rho_{k'_2 k_2} \otimes \cdots \otimes \rho_{k'_n k_n} \\
&\{(\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \Delta_\theta) \cdots (\mathbb{1} \otimes \Delta_\theta) \, \Delta_\theta(g)\} \, |k'_1\,,\, k'_2\,,\, \cdots\, k'_n\rangle \;.
\end{aligned} \tag{3.1.17}
$$

This equation carries the main content of twisting.

Clearly $U(g)$ also gives a representation of the twisted Hopf Algebra $\mathfrak{P}$.

3.2 Statistics of States

Let $|k_1, k_2\rangle_{s_0, a_0}$ denote a two-particle boson (fermion) state with definite momenta for each particle for $\theta_{\mu\nu} = 0$. For identical particles the state $|k_2, k_1\rangle_{s_0, a_0}$ is not an independent state and is related to $|k_1, k_2\rangle_{s_0, a_0}$, in the usual case, by the equivalence

$$
|k_1, k_2\rangle_{s_0, a_0} \sim |k_2, k_1\rangle_{s_0, a_0} \tag{3.2.1}
$$

since

$$
\begin{aligned}
&|k_2, k_1\rangle_{s_0, a_0} = \frac{1}{2}(|k_2, k_1\rangle \pm |k_1, k_2\rangle) = \pm |k_1, k_2\rangle_{s_0, a_0} \;, \\
&|k_1, k_2\rangle := |k_1\rangle \otimes |k_2\rangle \;.
\end{aligned}
$$

A statistics operator τ takes one vector of the equivalence class to the other. Given the statistics operator τ the vectors in the Hilbert space are identified by

$$
V = \tau V \;. \tag{3.2.2}
$$

The usual statistics operator τ_0 corresponding to (3.2.1) is just the flip operator

$$
\tau_0 |k_1, k_2\rangle = |k_2, k_1\rangle \tag{3.2.3}
$$

with

$$
|k_1, k_2\rangle_{s_0, a_0} = \frac{1}{2}(\mathbb{1} \pm \tau_0)|k_1, k_2\rangle \;.
$$

Compatibility of Lorentz invariance with statistics means that two vectors, V and W, in the Hilbert space which are identified by an equivalence relation should transform under the group action, to vectors which are again identified with each other according to the same relations. In other words, an equivalence class should transform into an equivalence class. Applying this to two-particle sector (with an obvious generalization to arbitrary number of particles) one can easily see that this is equivalent to the requirement that statistics operator defined by Eq.(3.2.2) commutes with the coproduct.

The immediate consequence of this is that the 'commutative' statistics operator τ_0 leads to statistics not compatible with the twisted Poincaré symmetry. This is due to the fact that the twisted coproduct Δ_θ is not cocommutative, i.e.

$$\tau_0 \Delta_\theta(g) = \tau_0(g_\alpha^{(1)} \otimes g_\alpha^{(2)}) \ = (g_\alpha^{(2)} \otimes g_\alpha^{(1)})\tau_0 \ = \ \Delta'_\theta(g)\tau_0 \ \neq \ \Delta_\theta(g)\tau_0 \tag{3.2.4}$$

where we have written $\Delta_\theta(g)$ in the Sweedler notation with a summation over α.

But using the operator of the twist, Eq.(2.0.7), and the definition of the twisted coproduct, Eq.(3.1.16), one can easily construct an appropriate deformation of the statistics operator. An evident solution, τ_θ, that commutes with Δ_θ is

$$\tau_\theta = F_\theta^{-1} \tau_0 F_\theta \ .$$

Note that

$$\tau_\theta = \tau_0 F_\theta^2 = \tau_0 \mathcal{R} \ ,$$

where $\mathcal{R} := F_\theta^2$ satisfies the following identity

$$\mathcal{R} \ \Delta_\theta \ = \ \Delta'_\theta \ \mathcal{R} \ .$$

$\mathcal{R}$ is called R-matrix of the twisted Poincaré group.

The two-particle momentum eigenstates with twisted statistics are thus [1]

$$|k_1, k_2\rangle_{s_\theta, a_\theta} = \frac{\mathbb{1} \pm \tau_\theta}{2} |k_1, k_2\rangle \ , \tag{3.2.5}$$

where we have to use a representation of F_θ on the Hilbert space. We denote it by $\mathcal{F}_\theta$ (cf. Eq.(2.0.8)):

$$\mathcal{F}_\theta |k_1, k_2\rangle = e^{-\frac{i}{2}\theta_{\mu\nu} k_1^\mu k_2^\nu} |k_1, k_2\rangle \ .$$

It follows that

$$|k_1, k_2\rangle_{s_\theta, a_\theta} \ = \ \pm e^{i\theta_{\mu\nu} k_2^\mu k_1^\nu} \, |k_2, k_1\rangle_{s_\theta, a_\theta} \ . \tag{3.2.6}$$

One can easily generalize this to a multi-particle state. Consider a three-particle state. Again comparing the transformation of the state $|k_1, k_2, k_3\rangle_{s_\theta, a_\theta}$ and $|k_3, k_2, k_1\rangle_{s_\theta, a_\theta}$ one finds out that they must be related by

$$|k_1, k_2, k_3\rangle_{s_\theta, a_\theta} \ = \pm e^{i\theta_{\mu\nu} k_3^\mu k_2^\nu} e^{i\theta_{\mu\nu} k_3^\mu k_1^\nu} e^{i\theta_{\mu\nu} k_2^\mu k_1^\nu} \ |k'_3, k'_2, k'_1\rangle_{s_\theta, a_\theta} \ . \tag{3.2.7}$$

So when we exchange two particles, we must commute the two labels past the neighboring labels to bring them to the desired place, and include the exponential factors as in (3.2.7) for each such permutation.

Generalizing τ_θ to transpositions of adjacent particles in $|k_1\rangle \otimes |k_2\rangle \otimes |k_3\rangle \equiv |k_1, k_2, k_3\rangle$ and using them to fully symmetrize (antisymmetrize) $|k_1, k_2, k_3\rangle$, we can explicitly write $|k_1, k_2, k_3\rangle_{s_\theta, a_\theta}$.

[1] We use the same notation, τ_θ, for the statistics operator and its representation on the Hilbert space.

3.3 Scalar Product

Now we must chose a scalar product for multi-particle states which is compatible with the above statistics (and hence in turn compatible with the Lorentz group in the sense that the operators $U(g)$ are unitary in this scalar product).

First consider the two-particle states. A scalar product compatible with the above statistics (upto multiplicative constants) is

$$\begin{aligned}&(|k_1, k_2\rangle_{s_\theta,a_\theta}\ ,\ |p_1, p_2\rangle_{s_\theta,a_\theta}) = \\ &\quad 4k_1^0 k_2^0\,[\ \delta^3(k_1 - p_1)\delta^3(k_2 - p_2) \pm\ e^{-i\theta_{\mu\nu}p_2^\mu p_1^\nu}\delta^3(k_1 - p_2)\delta^3(k_2 - p_1)\ ]\ .\end{aligned} \tag{3.3.1}$$

Now the right-hand side has the same symmetry properties as the left-hand side. Note that the phase factor which gives the correct symmetry can only be included in the second term because of the positivity of scalar product. Also this single phase factor gives the correct symmetry under the exchange of particles in both vectors.

Generalization to higher number of particles is simple. The first term which involves delta functions in the same ordering as in the vectors in the scalar products is without any phase factor. The delta functions in the other terms can all be obtained from permutations of the second momentum labels in the first term, and we include a phase factor like in (3.3.1), for each such permutation. Again as in (3.2.7), when we permute two labels which are not nearest neighbors, we must commute them past the neighbors and include the phase factor for each of such permutation. As an illustration we write the scalar product of three particle states.

$$\begin{aligned}(|p_1, p_2, p_3\rangle_{s_\theta,a_\theta}, |k_1, k_2, k_3\rangle_{s_\theta,a_\theta}) \ =\ & 2^3 p_1^0 p_2^0 p_3^0 [\delta^3(p_1 - k_1)\delta^3(p_2 - k_2)\delta^3(p_3 - k_3) \\ & \pm e^{-i\theta_{\mu\nu}k_3^\mu k_2^\nu}\delta^3(p_1 - k_1)\delta^3(p_2 - k_3)\delta^3(p_3 - k_2) \\ & \pm e^{-i\theta_{\mu\nu}k_2^\mu k_1^\nu}\delta^3(p_1 - k_2)\delta^3(p_2 - k_1)\delta^3(p_3 - k_3) \\ & + e^{-i\theta_{\mu\nu}k_2^\mu k_1^\nu} e^{-i\theta_{\mu\nu}k_3^\mu k_1^\nu}\delta^3(p_1 - k_2)\delta^3(p_2 - k_3)\delta^3(p_3 - k_1) \\ & + e^{-i\theta_{\mu\nu}k_3^\mu k_1^\nu} e^{-i\theta_{\mu\nu}k_3^\mu k_2^\nu}\delta^3(p_1 - k_3)\delta^3(p_2 - k_1)\delta^3(p_3 - k_2) \\ & \pm e^{-i\theta_{\mu\nu}k_3^\mu k_1^\nu} e^{-i\theta_{\mu\nu}k_3^\mu k_2^\nu} e^{-i\theta_{\mu\nu}k_2^\mu k_1^\nu}\delta^3(p_1 - k_3)\delta^3(p_2 - k_2)\delta^3(p_3 - k_1)]\ .\end{aligned} \tag{3.3.2}$$

Now that we know the scalar product we can define a vector $\langle k_1, k_2|$ dual to $|k_1, k_2\rangle$. Note that from the definition of the scalar product, the phase in the statistics of the dual vector has negative sign relative to the vector. i.e.,

$${}_{s_\theta,a_\theta}\langle k_1, k_2| \ = \pm e^{-i\theta_{\mu\nu}k_2^\mu k_1^\nu}\,{}_{s_\theta,a_\theta}\langle k_2, k_1|\ . \tag{3.3.3}$$

4. Quantum Generators for the Poincaré Group with Twisted Coproduct

We now give the explicit formulae for the quantum operators for the twisted Poincaré group in terms of creation and annihilation operators.

Hereafter in this paper, we focus on spin zero particles and fields.

4.1 Creation/Annihilation Operators

We define the creation operator $a_k^\dagger$ to be the operator which adds a particle with momentum k to the list of particles as usual. But there is an ambiguity as to whether the particle should be added to the left or to the right of the list. In the usual case the left and the right of the list has no inherent meaning because the two states got by either of the procedures transform in a similar manner under the Poincaré group and have the same symmetry properties with respect to the exchange of the new particle with any of the others already present. But now this is no longer the case. The two states transform differently under the twisted action of the Poincaré group (because of the non-cocommutativity of the coproduct). Hence we must make a choice.

We chose to define $a_k^\dagger$ to be the operator which adds a particle to the right of the particle list

$$a_k^\dagger|k_1, k_2, \cdots, k_n\rangle_{s\theta} \ = \ |k_1, k_2, \cdots, k_n, k\rangle_{s\theta} \ . \tag{4.1.1}$$

(Later we will discuss what would change if we define $a_k^\dagger$ the other way).

Applying $a_k^\dagger$ twice and using the statistics we can easily see that the $a_k^\dagger$'s have commutation relation

$$a_{k_2}^\dagger a_{k_1}^\dagger \ = \ e^{i\theta_{\mu\nu}k_2^\mu k_1^\nu} \, a_{k_1}^\dagger a_{k_2}^\dagger \ . \tag{4.1.2}$$

Now let us see what is the effect of a_k, the adjoint of $a_k^\dagger$, on a general state. As usual we can find it out by calculating the matrix elements of $a_k|k_1, k_2, \cdots, k_n\rangle_{s\theta}$ with other states. We have

$$\begin{aligned}
(|p_1, p_2, \cdots, p_{n-1}\rangle_{s\theta}, a_k|k_1, k_2, \cdots, k_{n-1}, k_n\rangle_{s\theta})& \\
&= (a_k^\dagger|p_1, p_2, \cdots, p_{n-1}\rangle_{s\theta}, |k_1, k_2, \cdots, k_{n-1}, k_n\rangle_{s\theta}) \\
&= (|p_1, p_2, \cdots, p_{n-1}, k\rangle_{s\theta}, |k_1, k_2, \cdots, k_{n-1}, k_n\rangle_{s\theta}) \ .
\end{aligned} \tag{4.1.3}$$

But using the scalar product in section (3.3), it is easily seen that this matrix element is the same as that of $|p_1, p_2, \cdots, p_{n-1}\rangle_{s\theta}$ with the state

$$\begin{aligned}
2k^0\delta^3(k-k_n)|k_1, k_2, \cdots, k_{n-1}\rangle_{s\theta} + 2k^0 e^{-i\theta_{\mu\nu}k^\mu k_n^\nu}\delta^3(k-k_{n-1})|k_1, k_2, \cdots, k_{n-2}, k_n\rangle_{s\theta}& \\
+ \cdots \ 2k^0 e^{-i\theta_{\mu\nu}k^\mu(k_n+k_{n-1}+\cdots+k_2)^\nu}\delta^3(k-k_1)|k_2, k_3, \cdots, k_n\rangle_{s\theta}& \ .
\end{aligned}$$

Since $|p_1, p_2, \cdots, p_{n-1}\rangle_{s\theta}$ is a general state, we have that

$$\begin{aligned}
a_k|k_1, k_2, \cdots, k_{n-1}, k_n\rangle_{s\theta} =& \\
&2k^0\delta^3(k-k_n)|k_1, k_2, \cdots, k_{n-1}\rangle_{s\theta} \\
&+ 2k^0 e^{-i\theta_{\mu\nu}k^\mu k_n^\nu}\delta^3(k-k_{n-1})|k_1, k_2, \cdots, k_{n-2}, k_n\rangle_{s\theta} + \cdots \\
&\cdots + 2k^0 e^{-i\theta_{\mu\nu}k^\mu(k_n+k_{n-1}+\cdots+k_2)^\nu}\delta^3(k-k_1)|k_2, k_3, \cdots, k_n\rangle_{s\theta} \ .
\end{aligned} \tag{4.1.4}$$

The commutator of a_k can be directly found by taking the adjoint of (4.1.2) , and we find

$$a_{k_2}a_{k_1} \ = \ e^{i\theta_{\mu\nu}k_2^\mu k_1^\nu} \, a_{k_1}a_{k_2} \ . \tag{4.1.5}$$

Now let us find the commutator of a_k and $a_k^\dagger$. Acting on a general state $|p_1, p_2, \cdots, p_n\rangle_{s\theta}$ by $a_{k_2}^\dagger a_{k_1}$, we get

$$\begin{aligned}
&a_{k_2}^\dagger a_{k_1}|p_1, p_2, \cdots, p_n\rangle_{s\theta} = \\
&\qquad 2k_1^0\delta^3(k_1 - p_n)|p_1, p_2, \cdots, p_{n-1}, k_2\rangle_{s\theta} \\
&\qquad + 2k_1^0 e^{-i\theta_{\mu\nu}k_1^\mu p_n^\nu}\delta^3(k_1 - p_{n-1})|p_1, p_2, \cdots, p_{n-2}, p_n, k_2\rangle_{s\theta} + \cdots \\
&\qquad \cdots + 2k_1^0 e^{-i\theta_{\mu\nu}k_1^\mu(p_n+p_{n-1}+\cdots+p_2)^\nu}\delta^3(k_1 - p_1)|p_2, p_3, \cdots, p_n, k_2\rangle_{s\theta} \ .
\end{aligned} \tag{4.1.6}$$

On the other hand acting by $a_{k_1} a_{k_2}^\dagger$ gives us

$$\begin{aligned}
&a_{k_1} a_{k_2}^\dagger|p_1, p_2, \cdots, p_n\rangle_{s\theta} = \\
&\qquad 2k_1^0\delta^3(k_1 - k_2)|p_1, p_2, \cdots, p_{n-1}, p_n\rangle_{s\theta} \\
&\qquad + 2k_1^0 e^{-i\theta_{\mu\nu}k_1^\mu k_2^\nu}\delta^3(k_1 - p_n)|p_1, p_2, \cdots, p_{n-1}, k_2\rangle_{s\theta} + \cdots \\
&\qquad + 2k_1^0 e^{-i\theta_{\mu\nu}k_1^\mu(p_n+k_2)^\nu}\delta^3(k_1 - p_{n-1})|p_1, p_2, \cdots, p_{n-2}, p_n, k_2\rangle_{s\theta} + \cdots \\
&\qquad \cdots + 2k_1^0 e^{-i\theta_{\mu\nu}k_1^\mu(p_n+p_{n-1}+\cdots+p_2+k_2)^\nu}\delta^3(k_1 - p_1)|p_2, p_3, \cdots, p_n, k_2\rangle_{s\theta} \ .
\end{aligned} \tag{4.1.7}$$

Dividing both sides of (4.1.7) by $e^{-i\theta_{\mu\nu}k_1^\mu k_2^\nu}$ and subtracting from (4.1.6), we get

$$a_{k_2}^\dagger a_{k_1} \;=\; e^{i\theta_{\mu\nu}k_1^\mu k_2^\nu}\, a_{k_1} a_{k_2}^\dagger \;-\; 2k_1^0\delta^3(k_1 - k_2) \ . \tag{4.1.8}$$

Or

$$a_{k_2} a_{k_1}^\dagger \;=\; e^{i\theta_{\mu\nu}k_1^\mu k_2^\nu}\, a_{k_1}^\dagger a_{k_2} \;+\; 2k_1^0\delta^3(k_1 - k_2) \ . \tag{4.1.9}$$

4.2 Transformation Law for Creation/Annihilation Operators

The transformation law for creation and annihilation operators is deduced from the transformation law of states. According to (3.1.16), we want to have

$$U(g) a_{k_2}^\dagger a_{k_1}^\dagger|0\rangle = e^{-\frac{i}{2}\theta_{\mu\nu}k_1^\mu k_2^\nu}\, e^{\frac{i}{2}\theta_{\mu\nu}(gk_1)^\mu(gk_2)^\nu}\, a_{gk_2}^\dagger a_{gk_1}^\dagger|0\rangle \ . \tag{4.2.1}$$

This can be achieved if $U(g)$ and $a_k^\dagger$ satisfy

$$U(g) a_k^\dagger = a_{gk}^\dagger e^{-\frac{i}{2}\theta_{\mu\nu}(gk)^\mu P^\nu}\, e^{\frac{i}{2}\theta_{\mu\nu}(k)^\mu(g^{-1})^\nu_\rho P^\rho} U(g) \ , \tag{4.2.2}$$

where g on the right hand side stands for the matrix of the Lorentz transformation and

$$P^\mu = \int \frac{d^3k}{2k^0} k^\mu a_k^\dagger a_k \ . \tag{4.2.3}$$

Notice that just like the usual momentum operator, P^μ obeys

$$[P^\mu, a_k^\dagger] = k^\mu a_k^\dagger \ , \tag{4.2.4}$$

$$[P^\mu, P^\nu] = 0 \ . \tag{4.2.5}$$

Hence we have

$$\begin{aligned}
&U(g)a^\dagger_{k_2}a^\dagger_{k_1}|0\rangle \\
&\quad = a^\dagger_{gk_2}e^{-\frac{i}{2}\theta_{\mu\nu}(gk_2)^\mu P^\nu}e^{\frac{i}{2}\theta_{\mu\nu}k_2^\mu((g^{-1})^\nu_\rho P^\rho)}a^\dagger_{gk_1}e^{-\frac{i}{2}\theta_{\mu\nu}(gk_1)^\mu P^\nu}e^{\frac{i}{2}\theta_{\mu\nu}k_1^\mu((g^{-1})^\nu_\rho P^\rho)}U(g)|0\rangle \\
&\quad = a^\dagger_{gk_2}a^\dagger_{gk_1}e^{-\frac{i}{2}\theta_{\mu\nu}(gk_2)^\mu(gk_1)^\nu}e^{\frac{i}{2}\theta_{\mu\nu}k_2^\mu((g^{-1})^\nu_\rho g^\rho_\sigma k_1^\sigma)}|0\rangle \\
&\quad = e^{\frac{i}{2}\theta_{\mu\nu}(gk_1)^\mu(gk_2)^\nu}e^{-\frac{i}{2}\theta_{\mu\nu}k_1^\mu k_2^\nu}a^\dagger_{gk_2}a^\dagger_{gk_1}|0\rangle
\end{aligned} \tag{4.2.6}$$

as required.

This works for any number of particles. We can write (4.2.2) as

$$U(g)a^\dagger_k = \rho_{k'k}(g^{(2)}_\alpha)a^\dagger_{k'}U(g^{(1)}_\alpha) \tag{4.2.7}$$

where $g^{(i)}_\alpha$ defined by coproduct (cf. Eq.(3.2.4)). Note the ordering of $g^{(2)}$ and $g^{(1)}$. Now we have

$$\begin{aligned}
&U(g)a^\dagger_{k_n}a^\dagger_{k_{n-1}}\cdots a^\dagger_{k_1}|0\rangle \\
&\quad =\rho_{k'_nk_n}(g^{(2)})\rho_{k'_{n-1}k_{n-1}}(g^{(1)(2)})\cdots\rho_{k'_1k_1}(g^{(1)(1)\cdots(2)}) \\
&\qquad a^\dagger_{k'_n}a^\dagger_{k'n_1}\cdots a^\dagger_{k_1}U(g^{(1)(1)\cdots(1)})|0\rangle \\
&\quad =\rho_{k'_nk_n}(g^{(2)})\rho_{k'_{n-1}k_{n-1}}(g^{(1)(2)})\cdots\rho_{k'_1k_1}(g^{(1)(1)\cdots(2)})a^\dagger_{k'_n}a^\dagger_{k'_{n-1}}\cdots a^\dagger_{k'_1}\epsilon(g^{(1)(1)\cdots(1)})|0\rangle \\
&\quad =a^\dagger_{k'_n}a^\dagger_{k'_{n-1}}\cdots a^\dagger_{k_1}(id\otimes\rho_{k'_1k_1}\otimes\rho_{k'_2k_2}\otimes\cdots\otimes\rho_{k'_nk_n}) \\
&\qquad (\epsilon\otimes id\otimes id\otimes\cdots\otimes id)(g^{(1)(1)\cdots(1)}\otimes g^{(1)(1)\cdots(2)}\otimes g^{(2)})|0\rangle \\
&\quad =a^\dagger_{k'_n}a^\dagger_{k'_{n-1}}\cdots a^\dagger_{k'_1}(\rho_{k'_1k_1}\otimes\rho_{k'_2k_2}\otimes\cdots\otimes\rho_{k'_nk_n}) \\
&\qquad (g^{(1)(1)\cdots(2)}\otimes g^{(1)(1)\cdots(2)}\cdots\otimes g^{(2)}|0\rangle \\
&\quad =\rho_{k'_1k_1}\otimes\rho_{k'_2k_2}\otimes\cdots\otimes\rho_{k'_nk_n} \\
&\qquad \{(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\Delta_\theta)\cdots(\mathbb{1}\otimes\Delta_\theta)\,\Delta_\theta(g)\}\,|k'_1\,,\,k'_2\,,\,\cdots\,k'_n\rangle_{s_\theta}
\end{aligned} \tag{4.2.8}$$

which is as desired. Here we have used

$$(\epsilon\otimes id)\Delta = id \tag{4.2.9}$$

where ϵ is the counit.

We can write eq (4.2.2) in a covariant way as (cf.(3.2.4))

$$U(g)a^\dagger_k = a^\dagger_{k'}(U\otimes\rho_{k'k})\Delta(g) \tag{4.2.10}$$

or

$$U(g)a^\dagger_k = a^\dagger_{k'}(\rho_{k'k}\otimes U)\Delta'(g)\ . \tag{4.2.11}$$

4.3 The Quantum Operators for the Poincaré group

Now we discuss the representation, $U(g)$, of the generators of the twisted Poincaré group on the Hilbert space of the theory. These generators must transform the multiparticle states according to the twisted coproduct and hence fulfil the eq.(4.2.2).

As discussed in [9] we have a map, "dressing transformation" [15–17], between usual creation/annihilation operators and the twisted ones given by

$$\begin{aligned} a_k^\dagger &= c_k^\dagger e^{\frac{i}{2}\theta_{\mu\nu}k^\mu P^\nu} \ , \\ a_k &= c_k e^{-\frac{i}{2}\theta_{\mu\nu}k^\mu P^\nu} \ , \end{aligned} \tag{4.3.1}$$

where c_k and $c_k^\dagger$ have the usual bosonic commutation relations. Note that the momentum operator has the same expression in terms of a_k's and c_k's :

$$P^\mu = \int \frac{d^3k}{2k^0} k^\mu a_k^\dagger a_k = \int \frac{d^3k}{2k^0} k^\mu c_k^\dagger c_k \ . \tag{4.3.2}$$

This is as expected since the twist does not change the coproduct of the momentum.

With the above expressions, $a_k^\dagger$ and a_k satisfy the correct commutation relations (4.1.2),(4.1.5),(4.1.8), and the states created by $a_k^\dagger$ have the correct scalar product. Hence, for any operator, we can replace a_k's and $a_k^\dagger$'s by the expressions in (4.3.1), to express it in terms of regular creation and annihilation operators c_k and $c_k^\dagger$'s.

To find the representation of the twisted Poincaré group, we will use as a guiding principle the result of [10,11], that one can use a representation of the commutative algebra on the noncommutative one to construct a representation of the twisted Poincaré group. The correct representation is just the usual one written in terms of the commutative algebra.[2] We will prove that the same is true in quantum case too. Namely, we will take as the representation the usual expressions for the *untwisted* Poincaré generators written in terms of c_k and $c_k^\dagger$'s. It is a trivial observation that they will satisfy the standard commutation relations as required. So the group structure is correct. Now we show that the same is true for the coproduct.

As was shown in the previous section, all we need to show is that $U(g)$'s acting on $a_k^\dagger$ satisfy (4.2.2). Since $U(g)$ is an untwisted generator, $c_k^\dagger$ transforms under its action in the usual way (cf. (4.2.2) for $\theta_{\mu\nu} = 0$):

$$U(g)c_k^\dagger = c_{gk}^\dagger U(g) \ . \tag{4.3.3}$$

P^μ, as all generators, has the usual expression in terms of $c_k^\dagger, c_k$'s. Hence we have

$$U(g)P^\mu = (g^{-1})^\mu_\nu P^\nu U(g) \ . \tag{4.3.4}$$

Using this, we have

$$\begin{aligned} U(g)a_k^\dagger =& U(g)c_k^\dagger e^{\frac{i}{2}\theta_{\mu\nu}k^\mu P^\nu} \\ =& c_{gk}^\dagger e^{\frac{i}{2}\theta_{\mu\nu}k^\mu((g^{-1})^\nu_\rho P^\rho)} U(g) \\ =& a_{gk}^\dagger e^{-\frac{i}{2}\theta_{\mu\nu}(gk)^\mu P^\nu} e^{\frac{i}{2}\theta_{\mu\nu}k^\mu((g^{-1})^\nu_\rho P^\rho)} U(g) \ , \end{aligned} \tag{4.3.5}$$

which fulfills Eq.(4.2.2).

[2]Note that for the generator of the momentum, this is trivially true, cf. eq.(4.3.2)

This completes the proof that the usual representation of the Poincaré generators, constructed out of untwisted creation/annihilation operators, c_k and $c_k^\dagger$, acting on the twisted Hilbert space realizes the representation of the *twisted* Poincaré group. This allows the discussion of twisted symmetries in a very simple and general way. Using this technique one can easily infer the transformation properties under C, P and T etc. Perhaps the full diffeomorphism symmetry can also be addressed in this manner.

As an illustration, consider the two particle state

$$|k_1, k_2\rangle_{s_\theta} = a^\dagger_{k_2} a^\dagger_{k_1}|0\rangle = e^{\frac{i}{2}\theta_{\mu\nu}k_2^\mu k_1^\nu} c^\dagger_{k_2} c^\dagger_{k_1}|0\rangle \ . \tag{4.3.6}$$

Now acting by $U(g)$, we just transform the $c^\dagger$'s to get,

$$\begin{aligned} U(g)|k_1, k_2\rangle_{s_\theta} &= e^{-\frac{i}{2}\theta_{\mu\nu}k_1^\mu k_2^\nu} c^\dagger_{gk_2} c^\dagger_{gk_1}|0\rangle & (4.3.7)\\ &= e^{-\frac{i}{2}\theta_{\mu\nu}k_1^\mu k_2^\nu} e^{+\frac{i}{2}\theta_{\mu\nu}gk_1^\mu gk_2^\nu} |gk_1, gk_2\rangle_{s_\theta} & (4.3.8)\\ &= \rho_{k_1' k_1} \otimes \rho_{k_2' k_2}(\Delta_\theta(g))|k_1', k_2'\rangle_{s_\theta} \ . & (4.3.9)\end{aligned}$$

as required.

5. On Invariant Interactions

The complete treatment of the construction of the invariant S-matrix is given in [10, 12]. There the precise conditions on the Hamiltonian are derived. Here we demonstrate how elementary is the proof of the uniqueness of the form of the Hamiltonian (in the sense that all fields should be multiplied with star-products) if one uses our result from the previous section on the representation of the twisted Poincaré group.

5.1 Fields

Let us begin with the introduction of quantum fields. This is done in complete analogy with the usual case. We define creation and annihilation fields from the creation and annihilation operators,

$$\begin{aligned} \Phi^+(x) &= \int d^3k e^{ik\cdot x} a_k^\dagger \ , \\ \Phi^-(x) &= \int d^3k e^{-ik\cdot x} a_k \end{aligned} \tag{5.1.1}$$

and the real Hermitian field

$$\Phi(x) = \Phi^+(x) + \Phi^-(x) \ . \tag{5.1.2}$$

Using eq (4.2.2), we find that the field $\Phi(x)$ obeys the following transformation law

$$U(g)\Phi(x) = \Phi(gx) e^{-\frac{1}{2}\theta_{\mu\nu}(g^\mu_\sigma \overleftarrow{\partial}^\sigma)P^\nu} e^{\frac{1}{2}\theta_{\mu\nu}\overleftarrow{\partial}^\mu (g^{-1})^\nu_\rho P^\rho} U(g) \ , \tag{5.1.3}$$

where $\overleftarrow{\partial}^\sigma$ acts to the left, only on the field argument, while P^μ is the total momentum operator and acts on everything to the right. But what is more important, the twisted field

$\Phi(x)$, (5.1.2), can be written in terms of untwisted field $\Phi_0(x)$, constructed as in (5.1.1) but with $c_k^\dagger$ and c_k. Using (4.3.1), we easily find

$$\Phi(x) = \Phi_0(x) e^{\frac{1}{2}\overleftarrow{\partial}^\mu \theta_{\mu\nu} P^\nu} \ . \tag{5.1.4}$$

Using this result, we see that the following is true

$$\Phi(x) * \Phi(x) = \left(\Phi_0(x)\Phi_0(x)\right) e^{\frac{1}{2}\overleftarrow{\partial}^\mu \theta_{\mu\nu} P^\nu} \tag{5.1.5}$$

where $*$ was defined in (2.0.8). This is a very important result, which makes the proof of the above statement about the interaction Hamiltonian almost trivial as we now demonstrate.

5.2 Interaction Hamiltonian

The first non-trivial term in the expansion of the S-matrix is

$$S^{(1)} = \int d^4x \, \mathcal{H}_{\mathcal{I}}(x) \ . \tag{5.2.1}$$

As we want to have a twisted invariant S-matrix, this term should be invariant separately. We claim that if $\mathcal{H}_{\mathcal{I}}(x)$ is a star-polynomial in $\Phi(x)$ with a typical term being

$$\mathcal{H}_I(x) = \Phi(x) * \Phi(x) * \cdots * \Phi(x) \tag{5.2.2}$$

then $S^{(1)}$ is twisted invariant. The proof is the combination of (5.1.5) and our construction of the generators of the twisted symmetry.

From (5.1.5) we have

$$\Phi(x)^n_* = \left(\Phi_0(x)^n\right) e^{\frac{1}{2}\overleftarrow{\partial}^\mu \theta_{\mu\nu} P^\nu} \ , \tag{5.2.3}$$

so the integration over 4-d (assuming that fields behave "nicely" at infinity) gives

$$\int d^4x \, \Phi(x)^n_* = \int d^4x \, \left(\Phi_0(x)^n\right) e^{\frac{1}{2}\overleftarrow{\partial}^\mu \theta_{\mu\nu} P^\nu} = \int d^4x \, \Phi_0(x)^n \ , \tag{5.2.4}$$

i.e., just a commutative result. Now we use the fact that the generators of the twisted symmetry are represented by the usual commutative generators constructed out of $c_k^\dagger$ and c_k (or equivalently, out of $\Phi_0(x)$). Then the invariance follows immediately.

By similar calculations, it is elsewhere shown to all orders that the S-operator is independent of $\theta_{\mu\nu}$ [4, 5, 10–12, 14].

As the S-operator is the same as the S-operator of the commutative theory (though the S-matrix is not), all trace of $\theta_{\mu\nu}$ is only in the statistics of in and out states.

It is worth noting that the whole structure developed so far is very rigid, in particular the S-operator is the only one which commutes with the symmetry operators giving us a twisted Poincaré invariant S-matrix.

All these considerations are valid only in the absence of gauge fields. Gauge field theories in our approach are discussed in [10–13].

6. On the Definition of Creation Operators

Here we comment on the ambiguity in the definition of creation/annihilation operators. As we saw in the section 3, we have to make a particular choice in the definition of creation operators, as to whether they add a particle to the left or to the right of the list of particles already present in the state. By now it should be clear what would happen if we had defined creation operators as adding the particle to the left of the list. The transformation properties and the statistics of the states of course do not depend on it as it was already fixed by the structure of the group. But the transformation of creation operators themselves and their commutation relations will be changed . Now they look like

$$U(g)a^\dagger_k = a^\dagger_{k'}(\rho_{k'k} \otimes U)\Delta(g), \tag{6.0.1}$$

$$a^\dagger_{k_2} a^\dagger_{k_1} = e^{-i\theta_{\mu\nu}k_2^\mu k_1^\nu}\, a^\dagger_{k_1} a^\dagger_{k_2}\ . \tag{6.0.2}$$

As a consequence, the invariant Hamiltonian now must be defined with a $*^{-1}$-product i.e., with a product defined by

$$\Phi(x) *^{-1} \Phi(x) = m_0 \mathcal{F}_\theta^{-1} \Phi(x) \otimes \Phi(y)\ . \tag{6.0.3}$$

That is why we adopted the other choice, as it gives the familiar $*$-product in the Hamiltonian. But the result that the S-operator is independent of $\theta_{\mu\nu}$ still holds. Hence this choice does not make any physical difference as the S-matrix of the theory is the same in both cases. In both cases , finally we have a theory with the same statistics for asymptotic states and the usual S-operator.

7. Conclusions

We have studied the general construction of field theories with twisted Poincaré invariance. A necessary consequence of such a symmetry is the twisting of statistics. The requirement of invariance of the S-matrix (and a convention regarding the definition of twisted creation and annihilation operators) forces one to choose the $*$-product between the fields in the interaction Hamiltonian. The final theory, in the most general case, consists of a theory with twisted statistics of the asymptotic states and an S-operator which is completely independent of the noncommutativity parameter $\theta_{\mu\nu}$.

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